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FORMULATION OF A TWO-SCALE MODEL OF TURBULENCE

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Abstract. A two-scale turbulence model is derived by averaging the two-point spectral evolution equation. In this model, the inertial range energy transfer and the dissipation rate can be unequal. The model is shown to reduce to a standard two-equation model in decaying turbulence.

Key words. multiple-scale turbulence model

Subject classification. Fluid Mechanics

1. Introduction. Despite the remarkable success of the two-equation turbulence model in predicting many practically important turbulent flows, it has some important shortcomings. For example, the linear eddy-viscosity relation between the Reynolds stress and the mean strain rate is inadequate in many problems; nonlinear eddy viscosity models, algebraic models, and finally the Reynolds stress transport model [1], have been developed in response to this problem.

Whereas these models focus their attention on the Reynolds stresses and leave the two-equation model itself basically intact, a complementary line of research [2], [3] has attempted to improve the two-equation model by addressing the over-simplification inherent in any description of turbulence by a single length-scale. It lead to the formulation of *multiple scale* models of turbulence in which the transport equations for turbulence kinetic energy and dissipation rate are each replaced by transport equations for the kinetic energy and dissipation rate pertaining to a definite range of scales of motion.

Multiple scale modeling attempts to treat the response of turbulence to changes in the large-scale motion more realistically than the two-equation model. Whereas the two-equation model assumes that the inertial range can adjust instantaneously to changes at the large scales, multiple scale modeling allows time-delays in this response and thereby permits a more refined picture of the time dependence of turbulence.

A representative multiple scale model is the model of Hanjalić, Launder, and Schiestel (HLS, [2]), which takes the form for homogeneous turbulence

$$\begin{aligned}
\dot{k}_p &= P - \epsilon_p \\
\dot{k}_t &= \epsilon_p - \epsilon_t \\
\dot{\epsilon}_p &= C_{p1}\frac{\epsilon_p}{k_p}P - C_{p2}\frac{\epsilon_p^2}{k_p} \\
\dot{\epsilon}_t &= C_{t1}\frac{\epsilon_p\epsilon_t}{k_t} - C_{t2}\frac{\epsilon_t^2}{k_t}
\end{aligned} \tag{1.1}$$

In this model, the turbulent fluctuations are partitioned into two regions identified by the subscripts p (production) and t (turbulence) of large- and small-scale fluctuations respectively. Otherwise, the standard notation is used in Eq. (1.1): k denotes turbulence kinetic energy, ϵ is the dissipation rate, P is production, and $C_{p1}, C_{p2}, C_{t1}, C_{t2}$ are model constants. Modeling the behavior of different scales of motion abandons

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the strictly single-point approach; it attempts a compromise between two-point models, which offer greater physical fidelity at the expense of greater computational complexity, and single-point modeling.

Early research on this type of model was frustrated by the inevitable appearance of a large number of model constants for which calibrating experiments could not be readily identified; thus, Eq. (1.1) requires four constants in place of the two constants of the comparable standard two-equation model. However, interest in multiple scale models has recently been revived by the possibility that numerical simulations could help identify the model constants [4]. At the same time, new approaches to multiple scale modeling based on two-point models have been advanced [3], [5], [6] which promise to eliminate, or at least reduce, the required empirical input.

The goal of the present work is to derive a two-scale model in which energy transfer and dissipation can be distinct and satisfy different transport equations. Kolmogorov's principle of *locality* of inertial range transfer [7] is used to separate these effects. Unlike the previous model of this type [6], the coefficients in the present model are independent of Reynolds number.

The multiple-scale viewpoint permits a fresh derivation of the two-equation model [3], [5], [6]. The two-equation model will be reconsidered from this viewpoint. Then a two-scale model is derived in which transfer through the large scales and the dissipation rate can be unequal.

Multiple-scale effects are expected to be important in turbulent flows dominated by disequilibrium between large and small scales. When such effects are absent, as they are in the standard self-similar flows used to calibrate turbulence models, the multiple-scale must reduce to a single-scale model with a unique similarity solution. This principle is applied to both the HLS model and the proposed model in the case of decaying turbulence. Conditions are found which prevent the existence of more than one exponent for power-law decay. In both cases, the condition is a simple inequality among the model constants.

The occurence of partitioned values of turbulence kinetic energy and dissipation rate naturally generates many new possibilities in relating the Reynolds stress and the strain rate; however, this issue will not be addressed in the present work.

2. The Two-equation Model as a Single-scale Model. The starting point is the isotropic part of the spectral evolution equation [9]

$$\dot{E}(\kappa) = T(\kappa) + \Pi(\kappa) - D(\kappa) + \mathcal{D}(\kappa) \tag{2.1}$$

where $E(\kappa)$ is the energy spectrum, $T(\kappa)$ is energy transfer spectrum, $\Pi(\kappa)$ is the production spectrum, $D(\kappa)$ is the dissipation spectrum, and $\mathcal{D}(\kappa)$ represents diffusion effects. Only homogeneous turbulence will be considered, so that $\mathcal{D}(\kappa) \equiv 0$. The dissipation spectrum is defined by

$$D(\kappa) = \nu\kappa^2 E(\kappa) \tag{2.2}$$

and closures must be provided for $T(\kappa)$ and $\Pi(\kappa)$.

The integrated quantities are

$$\begin{aligned} \int_0^\infty E(\kappa)d\kappa &= k \\ \int_0^\infty T(\kappa)d\kappa &= 0 \\ \int_0^\infty \Pi(\kappa)d\kappa &= P \\ \int_0^\infty D(\kappa)d\kappa &= \epsilon \end{aligned} \tag{2.3}$$

The second relation expresses the conservation of energy by nonlinear interaction. The energy equation follows from integrating Eq. (2.1) over all wavenumbers. Using Eqs. (2.3),

$$\dot{k} = P - \epsilon \tag{2.4}$$

This result is independent of the analytical form of $E(\kappa)$ and $\Pi(\kappa)$, and of the closure used to define $T(\kappa)$.

To find a second equation, it will be necessary to introduce some specific assumptions. The simplest steady solution of Eq. (2.1) is the *Kolmogorov steady state* defined by

$$E(\kappa) = \begin{cases} 0 & \text{if } \kappa \leq \kappa_0 \\ C_K \mathcal{T}^{2/3}\kappa^{-5/3} & \text{if } \kappa_0 \leq \kappa \leq \kappa_d \\ 0 & \text{if } \kappa \geq \kappa_d \end{cases} \tag{2.5}$$

corresponding to

$$T(\kappa) = -\mathcal{T}\{\delta(\kappa - \kappa_0) - \delta(\kappa - \kappa_d)\} \tag{2.6}$$

where $\mathcal{T}$ represents the energy transfer through the inertial range. In Eqs. (2.5)–(2.6), κ_0 is the integral scale, and $\kappa_d \sim (\epsilon/\nu^3)^{1/4}$ is the Kolmogorov scale. To maintain a steady state, production must balance transfer into the inertial range, and transfer out of the inertial range must balance viscous dissipation, so that

$$P = \epsilon = \mathcal{T} \tag{2.7}$$

More complex analytical forms for $E(\kappa)$ could be introduced, but this change adds nothing essential.

Now generalize Eq. (2.5) to the time-dependent case by letting $\epsilon = \epsilon(t)$ and $\kappa_0 = \kappa_0(t)$. For high Reynolds number turbulence, ignore the consequent evolution of κ_d can be ignored. Assume that the second equality from Eq. (2.7), namely

$$\epsilon = \mathcal{T} \tag{2.8}$$

continues to apply. Since

$$k = \frac{3}{2} C_K \epsilon^{2/3} \kappa_0^{-2/3} \tag{2.9}$$

it follows that

$$\dot{k} = C_K(\epsilon^{-1/3}\kappa_0^{-2/3}\dot{\epsilon} - \epsilon^{2/3}\kappa_0^{-5/3}\dot{\kappa}_0) \tag{2.10}$$

but this equation does not lead to the desired second equation directly, because it contatins the new unknown $\dot{\kappa}_0$. We consider two methods by which a second equation can be derived.

2.1. The method of Schiestel. The problem is solved in [3] by postulating that

$$\dot{\kappa}_0 \propto \frac{\epsilon}{E(\kappa_0)} \tag{2.11}$$

which is equivalent, for a Kolmogorov spectrum, to

$$\dot{\kappa}_0/\kappa_0 = \gamma\epsilon/k \tag{2.12}$$

Then Eqs. (2.10)–(2.12) give

$$\dot{k} = \frac{2}{3}\frac{k}{\epsilon}\dot{\epsilon} + \frac{2}{3}\gamma\epsilon \tag{2.13}$$

which can be re-arranged as

$$\dot{\epsilon} = \frac{3}{2}\frac{\epsilon}{k}P - (\frac{3}{2} + \gamma)\frac{\epsilon^2}{k} \tag{2.14}$$

with a rather good value for $C_{\epsilon 1}$ and a value for $C_{\epsilon 2}$ which depends on the choice of γ. This is essentially Eq. (27) of [3].

2.2. The method of moments. It can reasonably be objected that by assuming the length-scale equation Eq. (2.12) the problem has simply been transferred to another variable and therefore Eq. (2.14) has not been derived, but simply postulated indirectly.

An interesting alternative has been suggested in [5], [6]: by taking another moment of the spectral evolution equation Eq. (2.1), an additional equation is obtained and $\dot{\kappa}_0$ can be eliminated between this equation and Eq. (2.10). This implies a reduction of Eq. (2.1) to a finite-dimensional system by a Galerkin approximation.

For example, multiply Eq. (2.1) by κ^{-1} and integrate. With the assumptions Eqs. (2.5)–(2.6),

$$\begin{aligned} \frac{d}{dt}\Big[\int_0^\infty d\kappa\ \frac{E(\kappa)}{\kappa}\Big] &= \frac{d}{dt}\Big[\frac{3}{5}C_K\epsilon^{2/3}\kappa_0^{-5/3}\Big] \\ &= \frac{\dot{k}}{\kappa_0} - \frac{3}{5}C_k\epsilon^{-1/3}\kappa_0^{-5/3}\dot{\epsilon} \\ &= \frac{\dot{k}}{\kappa_0} - \frac{2}{5}\frac{k}{\epsilon\kappa_0}\dot{\epsilon} \end{aligned} \tag{2.15}$$

and

$$\int_0^\infty d\kappa\ \frac{T(\kappa)}{\kappa} = -\frac{\epsilon}{\kappa_0} + \frac{\epsilon}{\kappa_d} \approx -\frac{\epsilon}{\kappa_0} \tag{2.16}$$

To make the calculation definite, assume for the production spectrum

$$\Pi(\kappa) = \begin{cases} 0 & \text{if } \kappa \le \kappa_0 \\ \frac{4}{3}C_D\epsilon^{1/3}\kappa^{-7/3}S^2 & \text{if } \kappa_0 \le \kappa \le \kappa_d \\ 0 & \text{if } \kappa \ge \kappa_d \end{cases} \tag{2.17}$$

where S is the mean strain rate. Eq. (2.17) is consistent with the usual two-equation model, since integration over κ leads to

$$P = C_D\epsilon^{1/3}\kappa_0^{-4/3}S^2 = \frac{4}{9}\frac{C_D}{C_K^2}\frac{k^2}{\epsilon}S^2 = C_\nu\frac{k^2}{\epsilon}S^2 \tag{2.18}$$

which, for the values of inertial range constants recommended by Yakhot and Orszag [8] $C_D \approx 0.5, C_K \approx 1.6$, gives

$$C_\nu = \frac{4}{9}\frac{C_D}{C_K^2} \approx \frac{4}{45} \tag{2.19}$$

With this choice,

$$\int_0^\infty d\kappa\ \frac{\Pi(\kappa)}{\kappa} = \frac{4}{7}C_D\epsilon^{1/3}\kappa_0^{-7/3}S^2 = \frac{4}{7}\frac{P}{\kappa_0} \tag{2.20}$$

Note that

$$\int_0^\infty d\kappa\,\frac{D(\kappa)}{\kappa} \sim \nu\kappa_d^{2/3} \sim Re^{-1/2} \tag{2.21}$$

is negligible in the high Reynolds number limit.

Combining the results Eqs. (2.15), (2.16), (2.20), and (2.21),

$$\frac{1}{\kappa_0}(\dot{k} - \frac{2}{5}\frac{k\dot{\epsilon}}{\epsilon}) = -\frac{\epsilon}{\kappa_0} + \frac{4}{7}\frac{P}{\kappa_0} \tag{2.22}$$

consequently,

$$\dot{\epsilon} = \frac{5}{2}\frac{3}{7}\frac{\epsilon}{k}P \tag{2.23}$$

with a value $C_{\epsilon 1} = 15/14$ which is somewhat too small, but also $C_{\epsilon 2} = 0$.

The absence of a destruction term comes about because of the cancellation of $\dot{k}$ and ϵ in Eq. (2.22). This cancellation proves to be independent of the order of the moment taken, provided of course that the integrals converge at large κ.

The question then arises whether the result of Eq. (2.16) might not depend strongly on the assumptions made and whether more generally, we should not obtain

$$\int_0^\infty d\kappa\,\frac{T(\kappa)}{\kappa} = -C_T\frac{\epsilon}{\kappa_0} \tag{2.24}$$

with $C_T \neq 1$. But Eq. (2.16) is reproduced exactly if the closure for $T(\kappa)$ given by Eq. (2.6) is replaced by Leith diffusion closure for $T(\kappa)$ [9],

$$T(\kappa) = -c_1\frac{\partial}{\partial\kappa}\kappa^2\sqrt{\kappa E(\kappa)}E(\kappa) + c_2\frac{\partial}{\partial\kappa}\kappa^3\sqrt{\kappa E(\kappa)}\frac{\partial E}{\partial\kappa} \tag{2.25}$$

with constants compatible with the existence of Kolmogorov and equipartition spectra [9]. The evaluation of the moment of order -1 has been proposed [5] for the EDQNM energy transfer model.

To conclude, write the ϵ transport equation in the form

$$\dot{\epsilon} = C_{\epsilon 1}\frac{\epsilon}{k}P - (C_T - 1)\frac{\epsilon^2}{k} \tag{2.26}$$

The values of $C_{\epsilon 1}$ and C_T are subjects for future investigation.

2.3. The moment equation of order +2. It is natural to attempt to derive an equation for viscous dissipation by multiplying Eq. (2.1) by $\nu\kappa^2$ and integrating. As noted in [6], this approach has the very attractive feature that the divergences [10] of order $Re^{1/2}$ cancel, for

$$\int_0^\infty \kappa^2 T(\kappa) = \epsilon\kappa_d^2 \tag{2.27}$$

and

$$-\int_0^\infty d\kappa\,\kappa^2 D(\kappa) = -\nu\epsilon^{2/3}\kappa_d^{10/3} \tag{2.28}$$

and the definition of κ_d shows that the right sides of Eqs. (2.27) and (2.28) can sum to zero. However,

$$\int_0^\infty d\kappa\,\nu\kappa^2\Pi(\kappa) \sim \nu\epsilon^{1/3}S^2\kappa_d^{2/3} \tag{2.29}$$

is of order $Re^{-1/2}$ and the remaining contribution from the transfer term, $-\nu\kappa_0^2\epsilon$ is of order Re^{-1}. Thus, unless other contributions can be found, the right side of the moment equation of order two vanishes in the limit of infinite Re. Thus, although the moment equation of order -1 does not provide satisfactory model constants, it generates a more satisfactory model form than does the moment equation of order +2.

3. Multiple-scale Models. The simplest generalization of the previous single-scale model results from from taking the piecewise-Kolmogorov spectrum

$$E_i(\kappa) = \epsilon_i^{2/3}\kappa^{-5/3} \text{ for } \kappa_{i-1} \le \kappa \le \kappa_i \text{ with } i \ge 1 \tag{3.1}$$

and

$$T(\kappa) = -\sum_{1\le i\le n} \epsilon_i[\delta(\kappa-\kappa_{i-1}) - \delta(\kappa-\kappa_i)] \tag{3.2}$$

Eq. (3.2) defines a *shell model* of turbulence, in which energy is transferred from each discrete region of scales to the adjacent region of smaller scales. This picture greatly oversimplifies the actual energy transfer process, which is mediated by triad interactions and which permits both forward and backward energy transfer and transfer between non-adjacent regions. A more accurate description is given by two-point closures like DIA and EDQNM (Laporta, 1995).

Set corresponding to Eq. (2.17),

$$\Pi_i(\kappa) = \frac{4}{3}C_D\epsilon_i^{1/3}\kappa^{-7/3}S^2 \quad \text{for } \kappa_{i-1} \le \kappa \le \kappa_i \tag{3.3}$$

Then the moments of order zero, obtained by integrating over the regions $\kappa_{i-1} \le \kappa < \kappa_i$, give partial energy equations

$$\dot{k}_i = \epsilon_{i-1} - \epsilon_i + P_i - D_i \tag{3.4}$$

where

$$\begin{aligned} P_i &= C_D\epsilon_i^{1/3}(\kappa_{i-1}^{-4/3} - \kappa_i^{-4/3}) \\ D_i &= \frac{3}{4}\nu C_K(\kappa_i^{4/3} - \kappa_{i-1}^{-4/3}) \end{aligned} \tag{3.5}$$

and by definition, $\epsilon_0 = 0$. Unlike the single-scale energy equation Eq. (2.4), Eq. (3.4) depends on the special hypotheses made in Eqs. (3.1)–(3.2).

Eq. (3.1) implies

$$k_i = \frac{3}{2}C_K\epsilon_i^{2/3}(\kappa_{i-1}^{-2/3} - \kappa_i^{-2/3}) \tag{3.6}$$

from which differentiation in time gives

$$\dot{k}_i = C_K\epsilon_i^{-1/3}(\kappa_{i-1}^{-2/3} - \kappa_i^{-2/3})\dot{\epsilon}_i - C_K\epsilon_i^{2/3}\kappa_{i-1}^{-5/3}\dot{\kappa}_{i-1} + C_K\epsilon_i^{2/3}\kappa_i^{-5/3}\dot{\kappa}_i \tag{3.7}$$

As in the previous section, another relation is required to eliminate the new quantities $\dot{\kappa}_i$, and we can proceed either by following Schiesel's approach of introducing a new differential equation for $\dot{\kappa}_i$ or by forming moments of the the dynamic equations in each spectral region.

3.1. Schiestel's method. There are several natural choices to close $\dot{\kappa}_i$ in terms of local turbulence quantities, each of which will lead to a different model. One natural generalization of Eq. (2.12) to multiple regions is to set

$$\dot{\kappa}_i = \gamma\kappa_i{\epsilon_i}^{1/3}\kappa_i^{2/3} \tag{3.8}$$

so that the ratio $\dot{\kappa}_i/\kappa_i$ is determined by the local Kolmogorov frequency.

Schiestel ([3], Eq. (24)) suggests instead that $\dot{\kappa}_i$ be determined as the ratio of the flux into the region $\kappa_i \leq \kappa \leq \kappa_{i+1}$ divided by the local energy density at κ_i. In the present notation, this leads to

$$\dot{\kappa}_i = \gamma \frac{\epsilon_i}{E_{i+1}(\kappa_i)} = \gamma \frac{\epsilon_i}{C_k \epsilon_{i+1}^{2/3}} \kappa_i^{5/3} \tag{3.9}$$

where the choice of E_{i+1}, the energy density for scales greater than κ_i seems to be the most natural.

Substituting Eq. (3.9) in Eq. (3.7) leads to

$$\dot{k}_i = \frac{2}{3} \frac{k_i}{\epsilon_i} \dot{\epsilon}_i - \gamma \epsilon_{i-1} + \gamma \epsilon_i^{2/3} \frac{\epsilon_i}{\epsilon_{i+1}^{2/3}} \tag{3.10}$$

which can be rearranged as the partial ϵ equation

$$\begin{aligned} \dot{\epsilon}_i &= \frac{3}{2} \frac{\epsilon_i}{k_i} (P_i - \epsilon_i) + \frac{3}{2} \gamma \frac{\epsilon_{i-1} \epsilon_i}{k_i} - \frac{3}{2} \gamma \frac{\epsilon_i^{2/3}}{\epsilon_{i+1}^{2/3}} \frac{\epsilon_i^2}{k_i} \\ &= \frac{3}{2} \frac{\epsilon_i}{k_i} P_i + \frac{3}{2} \gamma \frac{\epsilon_{i-1} \epsilon_i}{k_i} - \frac{3}{2} \frac{\epsilon_i^2}{k_i} [1 + \gamma \frac{\epsilon_i^{2/3}}{\epsilon_{i+1}^{2/3}}] \end{aligned} \tag{3.11}$$

This result differs slightly from Eq. (25) of [3] because of the final term which modifies the destruction term. P_i can be neglected except in the regions with small i and D_i can be neglected except in the regions with large i; in the intermediate regions, the energy content is determined by nonlinear energy transfer alone. It is therefore a reasonable approximation to set $P_i = 0$ for $i \geq 2$ and $D_i = 0$ for $i < n$. Then the production of ϵ_i comes primarily from the term proportional to ϵ_{i-1}, the flux into the region $\kappa \geq \kappa_i$ from the region $\kappa \leq \kappa_i$. With these approximations, Eq. (3.11) reduces to the transfer equations of the HLS model of Eq.(1.1) except for the modification noted above of the destruction term.

3.2. The method of moments. As in the derivation of the single-scale model, it can be objected that the approach of [3] requires some arbitrary choices, like Eq. (3.9). A natural generalization of the method of moments is to form the partial moment equations

$$\frac{d}{dt} \Big[\int_{\kappa_{i-1}}^{\kappa_i} d\kappa \, \frac{E_i(\kappa)}{\kappa} \Big] = \int_{\kappa_{i-1}}^{\kappa_i} d\kappa \, \frac{T_i(\kappa)}{\kappa} + \int_{\kappa_{i-1}}^{\kappa_i} d\kappa \, \frac{\Pi_i(\kappa)}{\kappa} - \int_{\kappa_{i-1}}^{\kappa_i} d\kappa \, \frac{D_i(\kappa)}{\kappa} \tag{3.12}$$

With the piecewise Kolmogorov forms Eqs. (3.1), (3.2), and (3.3), this gives

$$\begin{aligned} &\frac{d}{dt} \Big[C_K \frac{3}{5} \epsilon_i^{2/3} (\kappa_{i-1}^{-5/3} - \kappa_i^{-5/3}) \Big] = \\ &\frac{\epsilon_{i-1} - \epsilon_i}{\kappa_{i-1}} + \frac{4}{7} C_D \epsilon_i^{1/3} (\kappa_{i-1}^{-7/3} - \kappa_i^{-7/3}) S^2 - 3\nu C_K (\kappa_i^{1/3} - \kappa_{i-1}^{1/3}) \end{aligned} \tag{3.13}$$

where again, $\epsilon_0 = 0$. In integrating the delta functions in $T_i(\kappa)$, the limits are $\kappa_{i-1} \leq \kappa < \kappa_i$.

The elimination of the quantities κ_i is straightforward but leads to lengthy expressions. To illustrate the procedure, consider a two-scale model with partial energies k_1, k_2 and partial transfers ϵ_1, ϵ_2. Assume that ϵ_2 balances the viscous dissipation so that this model generalizes the usual two-equation model by allowing transfer and dissipation to be unequal: compare [6].

The partial energy equations are

$$\begin{aligned} \dot{k}_1 &= P_1 - \epsilon_1 - D_1 \\ \dot{k}_2 &= P_2 + \epsilon_1 - D_2 \end{aligned} \tag{3.14}$$

where in the second equation, integration over $\kappa \geq \kappa_1$ causes the contributions from ϵ_2 to cancel. Introduce the approximations discussed above,

$$
\begin{aligned}
P &\approx P_1 \\
P_2 &\approx 0 \\
D &\approx D_2 \\
D_1 &\approx 0
\end{aligned}
\tag{3.15}
$$

and to insure the overall energy balance

$$\dot{k}_1 + \dot{k}_2 = \dot{k} = P - D \tag{3.16}$$

set

$$D = \epsilon_2 \tag{3.17}$$

Then Eq. (3.14) becomes

$$
\begin{aligned}
\dot{k}_1 &= P - \epsilon_1 \\
\dot{k}_2 &= \epsilon_1 - \epsilon_2
\end{aligned}
\tag{3.18}
$$

Effectively, this is a model in which energy transfer through the large scales and the viscous dissipation can be unequal. However, the derivation does not require the formation of the moment of order +2.

Since

$$
\begin{aligned}
k_1 &= \frac{3}{2} C_K \epsilon_1^{2/3} (\kappa_0^{-2/3} - \kappa_1^{-2/3}) \\
k_2 &= \frac{3}{2} C_K \epsilon_2^{2/3} \kappa_1^{-2/3}
\end{aligned}
\tag{3.19}
$$

the scale ratio κ_1/κ_0 can be eliminated through

$$\frac{k_1}{k_2} = \left(\frac{\epsilon_1}{\epsilon_2}\right)^{2/3} \left[\left(\frac{\kappa_1}{\kappa_0}\right)^{2/3} - 1\right] \tag{3.20}$$

or

$$\frac{\kappa_1}{\kappa_0} = \left\{\frac{k_1}{k_2}\left(\frac{\epsilon_2}{\epsilon_1}\right)^{2/3} + 1\right\}^{3/2} \tag{3.21}$$

Note that Eq. (3.20) implies that $\kappa_1 \geq \kappa_0$ and that Eq. (3.19 implies that $\kappa_1 = \kappa_0$ is equivalent to $k_1 = 0$. The relations

$$\dot{k}_1 = \frac{2}{3}\frac{k_1}{\epsilon_1}\dot{\epsilon}_1 - C_K \epsilon_1^{2/3} \kappa_0^{-5/3} \dot{\kappa}_0 + C_K \epsilon_1^{2/3} \kappa_1^{-5/3} \dot{\kappa}_1 \tag{3.22}$$

$$\dot{k}_2 = \frac{2}{3}\frac{k_2}{\epsilon_2}\dot{\epsilon}_2 - C_K \epsilon_2^{2/3} \kappa_1^{-5/3} \dot{\kappa}_1 \tag{3.23}$$

can be used to eliminate $\dot{\kappa}_0$ and $\dot{\kappa}_1$.

It is easiest to begin the derivation with the second wavenumber partition. A calculation similar to that leading to the single-scale result Eq. (2.26) results in

$$-\frac{2}{5}\frac{k_2}{\epsilon_2}\dot{\epsilon}_2 + \dot{k}_2 = (\epsilon_1 - \epsilon_2) C_{T1} \tag{3.24}$$

where the present theory actually predicts $C_{T1}=1$. Even if this value is left general, in the notation of Eq. (1.1), this calculation predicts $C_{t1}=C_{t2}$. Eq. (3.24) can be rearranged as

$$\dot{\epsilon}_2 = \frac{5}{2}\frac{\epsilon_2}{k_2}(1-C_{T1})(\epsilon_1-\epsilon_2) \tag{3.25}$$

where a small production term proportional to P_2 has been neglected. Note that the production of ϵ_2 comes primarily from ϵ_1.

Straightforward calculation leads to the ϵ_1 equation

$$\begin{gathered}\Big\{\frac{2}{5}\frac{k_1}{\epsilon_1} - \frac{4}{15}\Big(1-\frac{\kappa_0}{\kappa_1}\Big)\frac{k_2}{\epsilon_1^{1/3}\epsilon_2^{2/3}}\Big\}\dot{\epsilon}_1 = \\ \frac{3}{7}P - (1-C_{T0})\epsilon_1 - \frac{2}{3}\Big(1-\frac{\kappa_0}{\kappa_1}\Big)\Big(\frac{\epsilon_1}{\epsilon_2}\Big)^{2/3}(1-\frac{5}{2}C_{T1})(\epsilon_1-\epsilon_2)\end{gathered} \tag{3.26}$$

Note that the last contribution arises from the $\dot{\kappa}_1$ term of Eq. (3.22), which is eliminated in terms of $\dot{\epsilon}_2$ through Eq. (3.23).

To summarize, the two-scale model contains the partial energy equations,

$$\begin{aligned}\dot{k}_1 &= P - \epsilon_1 \\ \dot{k}_2 &= \epsilon_1 - \epsilon_2\end{aligned} \tag{3.27}$$

and the partial dissipation equations,

$$\begin{gathered}\Big\{\frac{2}{5}\frac{k_1}{\epsilon_1} - \frac{4}{15}\Big(1-\frac{\kappa_0}{\kappa_1}\Big)\frac{k_2}{\epsilon_1^{1/3}\epsilon_2^{2/3}}\Big\}\dot{\epsilon}_1 = \\ \frac{3}{7}P - (1-C_{T0})\epsilon_1 - \frac{2}{3}\Big(1-\frac{\kappa_0}{\kappa_1}\Big)\Big(\frac{\epsilon_1}{\epsilon_2}\Big)^{2/3}(1-\frac{5}{2}C_{T1})(\epsilon_1-\epsilon_2)\end{gathered} \tag{3.28}$$

$$\dot{\epsilon}_2 = \frac{5}{2}\frac{\epsilon_2}{k_2}(1-C_{T1})(\epsilon_1-\epsilon_2) \tag{3.29}$$

with the definition

$$\frac{\kappa_1}{\kappa_0} = \Big\{\frac{k_1}{k_2}\Big(\frac{\epsilon_2}{\epsilon_1}\Big)^{2/3}+1\Big\}^{3/2} \tag{3.30}$$

This two-scale, four-equation model separates large- and small-scale transfer without requiring Reynolds number dependent coefficients. Deficiencies of the derivation include the failure to predict $C_{\epsilon 2}$ and the equality $C_{t1}=C_{t2}$ in the small-scale transfer equation Eq. (3.29).

4. Relaxation to the Single-scale Model. Similarity solutions have proven indispensible in calibrating turbulence models. Examples include decaying turbulence and spatially self-similar turbulent shear flows like jets, mixing layers, and wakes. All of these flows were applied to calibrate and validate the first mixing-length models of turbulence.

We must expect that when applied to a turbulent flow which relaxes to a self-similar evolution, any multiple-scale model must relax to a single-scale model with a unique similarity solution. In particular, the additional freedom allowed in multiple-scale models must not permit spurious results like multiple power-law exponents in decaying turbulence. Both the HLS model and the proposed model will next be analyzed in decaying turbulence, and the conditions which prevent the existence of multiple exponents are found. In both cases, the condition is a simple inequality among the model constants.

4.1. The HLS model. Consider decaying turbulence described by the HLS model Eq. (1.1) with $P = 0$. Look for a solution

$$k_p = a_p t^\alpha$$
$$k_t = a_t t^\alpha$$
$$\epsilon_p = e_p t^{\alpha-1}$$
$$\epsilon_t = e_t t^{\alpha-1} \tag{4.1}$$

Substituting in Eq. (1.1) leads to a system of homogeneous equations in the constants a_p, a_t, e_p, e_t which has a nontrivial solution provided

$$0 = \begin{vmatrix} \alpha & 0 & 1 & 0 \\ 0 & \alpha & -1 & 1 \\ \alpha-1 & 0 & C_{p2} & 0 \\ 0 & \alpha-1 & -C_{t1} & C_{t2} \end{vmatrix} = (\alpha C_{t2} - \alpha + 1)(\alpha C_{p2} a + 1) \tag{4.2}$$

The solutions are $\alpha = -1/(C_{t2} - 1)$ and $\alpha = -1/(C_{p2} - 1)$ and the corresponding amplitude ratios are easily found to be

$$\begin{array}{lcl} a_p : a_t = & 0 : 1 & \text{if } \alpha = -1/(C_{t2} - 1) \\ a_p : a_t = & C_{p2} - C_{t2} : C_{t2} - C_{t1} & \text{if } \alpha = -1/(C_{p2} - 1) \end{array} \tag{4.3}$$

The first solution in Eq. (4.3) obviously corresponds to the reduction of the two-scale model to a single-scale model since $k_p = 0$. To avoid the existence of a second power law in decaying turbulence, the second solution in Eq. (4.3) must be non-realizable or unstable; its non-realizability is assured if one of the amplitudes a_p or a_t must be negative. This occurs if

$$C_{p2} < C_{t2} \text{ and } C_{t1} < C_{t2} \tag{4.4}$$

The decay equations were integrated for models satisfying and violating the condition Eq. (4.4). The results are shown in Fig. (4.1). First, the model constants were arbitrarily chosen as $C_{p2} = 1.5, C_{t1} = 1.2, C_{t2} = 2.0$ and the initial conditions were $\kappa_p(0) = \kappa_t(0) = 0.1, \epsilon_p(0) = \epsilon_t(0) = 1.0$. In this case, Eq. (4.4) is satisfied. The resulting decay is shown in the left graph in Fig. (4.1). It shows that the energy k_p approaches zero after an initial transient, indicating that at long times, the multiple-scale model reduces to a single-scale model. The dotted line shows the power law decay $\kappa \sim t^{-1}$ expected in this case.

To demonstrate that Eq. (4.4) is needed because the second solution in Eq. (4.3) can be stable, the decay equations were integrated for a case which does not satisfy the constraint, $C_{p2} = 3.0, C_{t1} = 1.2, C_{t2} = 2.0$. The results are shown in the center and right-hand graphs in Fig. (4.1). In the center graph, the initial conditions were $\kappa_p(0) = 0, \kappa_t(0) = 0.1, \epsilon_p(0) = 0, \epsilon_t(0) = 1.0$ while in the right-hand graph, the initial conditions were $\kappa_p(0) = \kappa_t(0) = 0.1, \epsilon_p(0) = \epsilon_t(0) = 1.0$. Clearly, the power law decay is different in each case; the dotted line again corresponds to the power law $k \sim t^{-1}$. Thus, if the conditions expressed by Eq. (4.4) are not satisfied, two distinct time-scaling laws can exist for decaying turbulence.

4.2. The moment model. The analysis for the moment model Eqs. (3.27)–(3.30) is similar. It is simpler to integrate Eq. (3.13) including the empirical factors in the dissipation rate terms. Substituting the power-law decay Eq. (4.1) and the additional relations

$$\kappa_0 = b_0 t^{-\alpha/2-1}$$
$$\kappa_1 = b_1 t^{-\alpha/2-1} \tag{4.5}$$

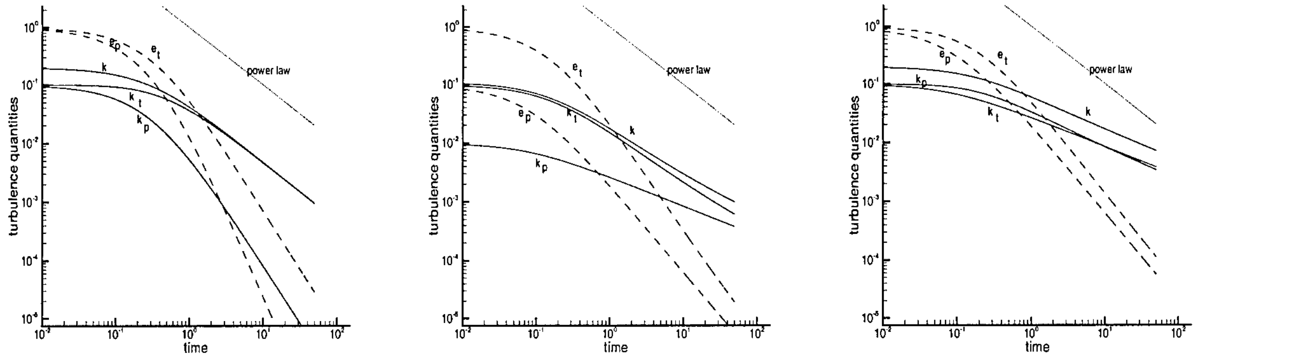

FIG. 4.1. *decaying turbulence, HLS model: (left) $C_{t1} < C_{p2} < C_{t2}$ one decay mode; (center) $C_{t1} < C_{p2} < C_{t2}$ two decay modes; (right) $C_{t1} < C_{t2} < C_{p2}$ two decay modes. The center and right figures show the possibility of two different power law decay rates when the condition in Eq. (4.4 is not satisfied.*

in Eq. (3.13),

$$\frac{3}{5}C_K e_1^{2/3}(b_0^{-5/3} - b_1^{-5/3})(\frac{3}{2}\alpha + 1) = -\frac{e_1}{b_0}C_{T1} \tag{4.6}$$

$$\frac{3}{5}C_K e_2^{2/3} b_1^{-5/3}(\frac{3}{2}\alpha + 1) = -\frac{e_1 - e_2}{b_1}C_{T2} \tag{4.7}$$

The consistency of these equations evidently requires

$$\kappa^{5/3} - 1 = \frac{C_{T1}}{C_{T2}}\kappa(\kappa^{2/3} - 1) \tag{4.8}$$

where $\kappa = \kappa_1/\kappa_0$. The only solution is $\kappa = 1$ corresponding to $\kappa_1 = \kappa_0$ provided $C_{T1} \leq C_{T2}$ but multiple solutions are possible once C_{T1} is large enough relative to C_{T2}. For example, a solution $\kappa \approx 1.2$ exists if $C_{T2} = 2C_{T1}$. Note that if $\kappa = 1$, then $k_1 = 0$, again indicating reduction of the multiple-scale model to a single-scale model.

In the case that only one solution exists, the power-law decay rate is again

$$\alpha = -\frac{1}{C_{\epsilon 2} - 1} \tag{4.9}$$

where

$$C_{\epsilon 2} = \frac{5}{2}(1 - C_{T1}) \tag{4.10}$$

indicating the reduction for decaying turbulence to the correct limit.

Examples of decaying turbulence computed with the moment model appear in Fig. (4.2). The initial conditions are

$$\begin{array}{lllll} \text{left graph Fig. (4.2):} & k_1(0) = 0.1 & k_2(0) = 0.1 & \epsilon_1(0) = 1.0 & \epsilon_2(0) = 1.0 \\ \text{right graph Fig. (4.2):} & k_1(0) = 0.15 & k_2(0) = 0.05 & \epsilon_1(0) = 3.0 & \epsilon_2(0) = 1.0 \end{array} \tag{4.11}$$

These initial conditions are chosen to correspond to the same initial conditions in a single-scale model, namely $k(0) = k_1(0) + k_2(0) = 0.2, \epsilon(0) = \epsilon_2(0) = 1.0$. At large times, both calculations follow the same power law decay, but there are clearly differences in the transient evolution before self-similarity is obtained. The decay of the total kinetic energy for the two sets of initial conditions is compared in Fig. (4.3).

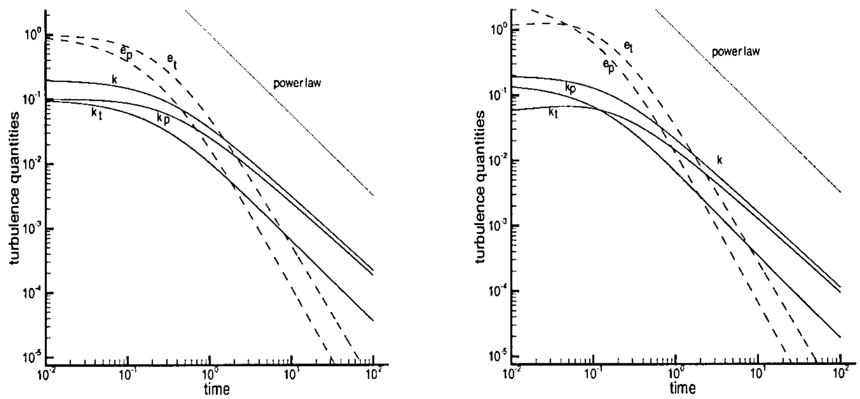

FIG. 4.2. *decaying turbulence, moment model: (left) initial conditions given by first of Eq. (4.11); (right) initial conditions given by second of Eq. (4.11). The graphs show the effects of different initial conditions on the decay of isotropic turbulence predicted by a two-scale model. The different sets of initial conditions correspond to the same initial conditions for a single-scale model*

Although both cases follow the same power law at long times, transient effects cause the energy to decay more quickly for the second set of initial conditions than for the first set. The capability to model this type of transient behavior illustrates the justification of multiple-scale modeling: a single-scale model could not distinguish between these two cases.

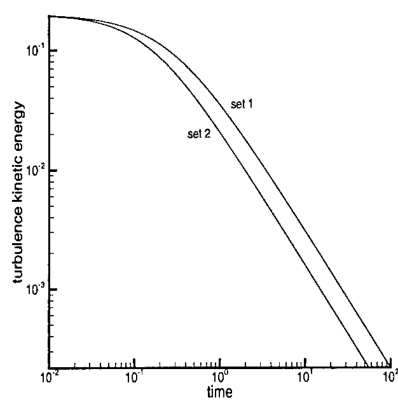

FIG. 4.3. *decaying turbulence, moment model, energy decay for the two sets of initial conditions in Eq. (4.11).*

5. Conclusions.

1. The arbitrary elements of the derivation of multiple scale models following [3] can be avoided by the method of moments, which reduces the continuous spectral evolution equations to a finite dimensional system.
2. The moment equation of order -1 leads to a well defined result, but the moment equation of order +2 is problematical.
3. However, the moment method leads to difficulties with the destruction of dissipation term both in

the single-scale model and in the multiple-scale models. The difficulty originates in the moment of the transfer integral.

4. A two scale model in which energy transfer through the large scales can be distinguished from viscous dissipation can be derived without forming the problematical moment of order +2.
5. Multiple-scale models must reduce to single-scale models when the turbulence evolution is self-similar. Conditions which insure this reduction are derived in the special case of decaying turbulence. The same analysis should be completed for other self-similar flows.

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